Exact & Closed Form Solutions For The Kolmogorov-Petrovsky-Piskunov-Fisher Equation in Cartesian and Polar Coordinates

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ABSTRACT

The Kolmogorov–Petrovsky–Piskunov–Fisher (KPP-Fisher) equation is a nonlinear reaction-diffusion partial differential equation (PDE) with significant applications in population genetics, ecology, and combustion theory. The KPP-Fisher equation for a distribution combines a linear diffusion term with a nonlinear reaction term which is independent of the derivatives of the distribution. The standard analytical technique for solving the involves transforming the equation into a nonlinear ordinary differential equation by a change of variables and analyzing the resulting wave form. We will demonstrate that an alternative approach inspired by non-classical symmetries but only using elementary methods leads to exact and closed form solutions for certain choices of boundary conditions. We will exhibit this result with quadratic and cubic sources and in Cartesian and polar coordinates.

PURPOSE

The purpose of this investigation is to explore the existence of closed-form solutions using elementary PDE techniques and to showcase how small modifications to the Fisher-KPP equation can disrupt exact solutions.

INTRODUCTION

- The Kolmogorov–Petrovsky–Piskunov–Fisher (KPP-Fisher) is a reaction-diffusion equation which describes the behavior of a beneficial evolutionary change over space and time [1].
- The KPP-Fisher equation is a nonlinear PDE which is central to modeling biological invasions, population dynamics, and neural activity [2].
- Much of the literature that looks into the Fisher-KPP equation is solved within the Cartesian coordinate system [1-5].
- The current research investigates the KPP-Fisher equation or extended KPP-Fisher equation in the cartesian coordinate system with alterations to the techniques used to solve for the solution [1-5].
- Integrating in different coordinate systems or employing Laplace transformations could yield new solution pathways, particularly for cases involving initial and boundary conditions that are hard to deal with by traditional methods.

METHODS

- $u(r,t)$ is the population density at radius r and time t
- r is the radial coordinate (distance from center)
- t is time
- u_t is the time derivative of u (describes rate of change over time)
- u_r and u_{rr} is the spatial derivatives in the radial direction
- D is the diffusion coefficient (describes rate of spatial spread)
- $\Delta u = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ is the Laplacian of u in polar (diffusion in radial symmetry)
- k is the intrinsic growth rate

METHODS

- Starting point for our 3 equations:
  - $u_t - D\Delta u = ku(1-u) \qquad (\alpha)$
  - $u_t - D\Delta u = ku(1-u^2) \qquad (\beta)$
  - $u_t - D\Delta u = ku^2(1-u) \qquad (\gamma)$
- Assume radial symmetry in Δu where $\Delta u = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$:
  - $u_t - D\left[u_{rr} + \frac{1}{r}u_r\right] = ku(1-u) \qquad (\alpha)$
  - $u_t - D\left[u_{rr} + \frac{1}{r}u_r\right] = ku(1-u^2) \qquad (\beta)$
  - $u_t - D\left[u_{rr} + \frac{1}{r}u_r\right] = ku^2(1-u) \qquad (\gamma)$
- With the boundary conditions:

$$\lim_{t\to\infty} u = 1 \qquad \lim_{t\to-\infty} u = 0 \qquad \lim_{r\to\infty} u = 0$$

- We solve using separation of variables, set each side to an arbitrary constant, and obtain the equations:

$$\frac{\dot{T}}{T} - k(1-RT) = C \qquad (\alpha.1)$$

$$\frac{\dot{T}}{T} - k(1-R^2T^2) = C \qquad (\beta.1)$$

$$\frac{\dot{T}}{T} - kRT(1-RT) = C \qquad (\gamma.1)$$

- With $(\alpha.2)$, $(\beta.2)$, $(\gamma.2)$ all equaling:

$$D\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right] = C \qquad (\alpha.2)\ \&\ (\beta.2)\ \&\ (\gamma.2)$$

- Where through algebraic manipulation $(\alpha.1)$ and $(\beta.1)$ become Bernoulli's Equations:

$$\frac{\dot{T}}{T^2} = (k+C)\frac{1}{T} - kR \qquad (\alpha.1)$$

$$\frac{\dot{T}}{T^3} = (k+C)\frac{1}{T^2} - kR^2 \qquad (\beta.1)$$

- $(\gamma.1)$ becomes a non-linear differential equation:

$$\dot{T} - CT - kRT^2 + kR^2T^3 = 0 \qquad (\gamma.1)$$

- $(\alpha.2), (\beta.2),$ and $(\gamma.2)$ become Helmholtz Equations:

$$R'' + \frac{1}{r}R' - \frac{C}{D}R = 0 \qquad (\alpha.2)\ \&\ (\beta.2)\ \&\ (\gamma.2)$$

RESULTS

After solving we get:

$$U = \frac{R(k+C)}{C_3 e^{-(k+C)t} + Rk} \qquad (\alpha)$$

$$U = \sqrt{\frac{R^2(k+C)}{kR^2 + (k+C)C_1 e^{-2(k+C)t}}} \qquad (\beta)$$

$$A\ln|T| + B\ln|T - T^\star| + C\ln|T - T^\dagger| = t + C_1 \qquad (\gamma.1)$$

$$\text{If } C \neq 0, \text{ then } \mathrm{R} = A_1 J_0\left(\sqrt{\frac{C}{D}}\,r\right) + B_1 Y_0\left(\sqrt{\frac{C}{D}}\,r\right) \qquad (\gamma.2)$$

$$\text{If } C = 0, \text{ then } \mathrm{R} = A_1 \ln r + B_1 \qquad (\gamma.2)$$

RESULTS

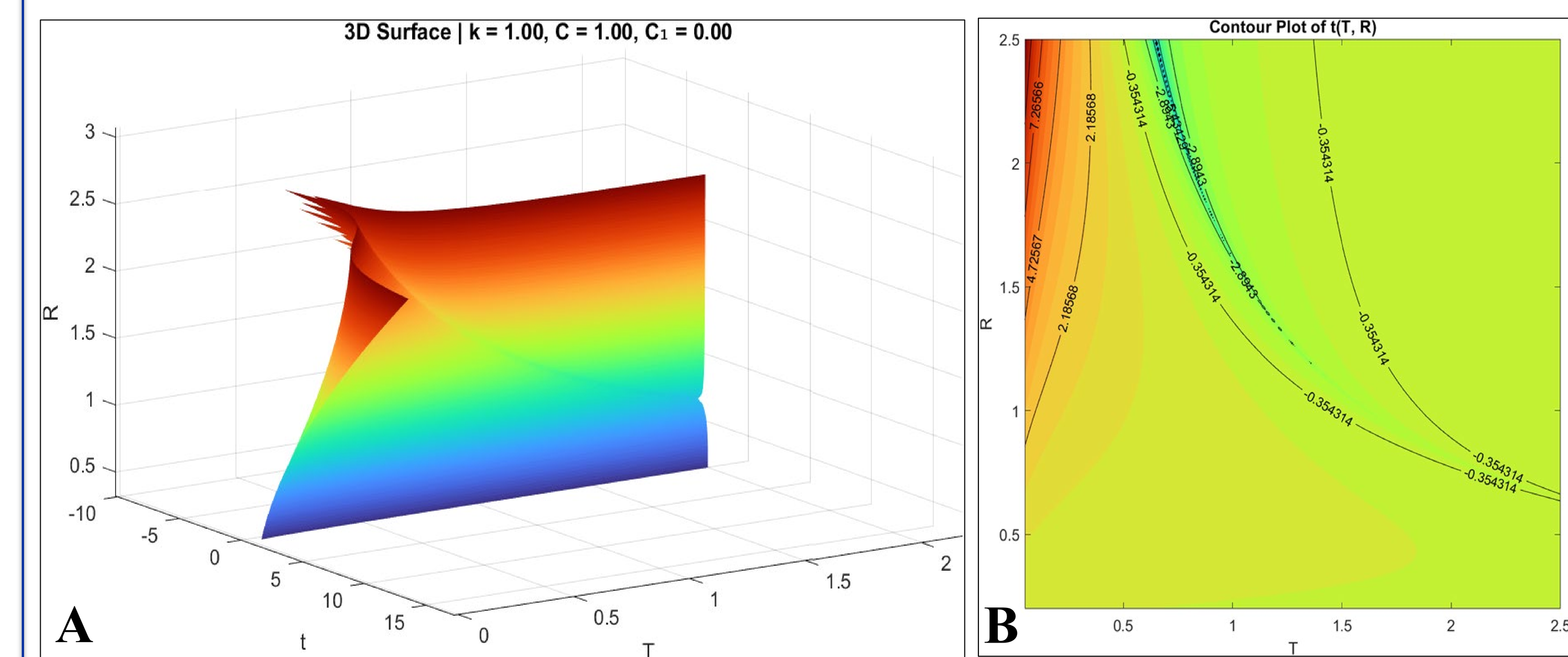

Figure 1. A) Plot of t, T, and R. B) Contour plot of time in relation to T and R.

DISCUSSION

- Reaction-Diffusion equations are used to model various phenomena in real-world situations and hypothetical scenarios.
- Their applications are used in several fields from mathematics and physics to biology and geology [2].
- This study developed a novel method for obtaining an exact solution to the KPP-Fisher equation in polar coordinates using various partial differential equation techniques.
- The results obtained offer a new mathematical tool and further deepens our understanding of nonlinear diffusion phenomena.
- Further study could be done on expanding to other extensions of the Fisher-KPP equation using the same coordinate system.

CONCLUSIONS

- Closed-form solutions were obtained using elementary techniques.
- Sensitivity to slight variations in the Fisher-KPP could result in non-exact solutions.
- Could be applied on other extensions of the Fisher-KPP to see if agreeable.
- The results will have a substantial impact on a variety of different fields such as the biology, conservation, and engineering.

REFERENCES

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