Computing Mixed and Pure Strategy Nash Equilibria of $m \times n$ Non-cooperative Bimatrix Games with Maple

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Abstract

Computing all the Nash equilibria of a non-cooperative bimatrix game by hand is cumbersome, if not impossible. For this reason, the development of computer programs to solve bimatrix games has become an integral part of the study of game theory. In this paper we present a Maple package composed of two procedures, ***nashpm*** and ***payoff***. When the user inputs an $m \times n$ bimatrix with real entries into ***nashpm***, all the mixed and pure Nash equilibria of the game are outputted. When the user inputs the Nash equilibria solutions into the ***payoff*** procedure, the corresponding expected payoffs for each player are outputted. Additionally, our package can calculate mixed and pure strategy Nash equilibria of bimatrix games with symbolic entries.

Introduction

Research focus: non-cooperative bimatrix games

- Two selfish/greedy players
- Each player has a finite number of pure strategies
- Players move simultaneously

Nash equilibrium- a set of strategies such that each player is playing a best response, given the strategies of all other players

Existing algorithms and computer programs:

- Computing one Nash equilibrium
  - Lemke-Howson algorithm [1]
- Computing all Nash equilibria
  - Lexicographic reverse search algorithm [2]
  - Enumeration of extreme equilibria algorithm [2]
  - Gambit [3]
- Approximating Nash equilibria
  - Method by Tsaknakis and Spirakis (0.3393-approximate, polynomial time) [4]
- Computing Nash equilibria with symbolic entries
  - Maple package by Wang, Ahmed, and Gutierrez [5]

The purpose of this study is to develop a Maple package that finds all the mixed and pure strategy Nash equilibria of non-cooperative $m \times n$ bimatrix games with numerical or symbolic entries. Additionally, we require the package to handle games with irrational entries and to output Nash equilibria in exact forms. Our Maple package can be an essential tool for researchers in game theory and game theory students when analyzing games of relatively large sizes.

Theory

Suppose we have the bimatrix game

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \\ s_m \end{array}\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & & t_n \end{array} \\ \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1n}, b_{1n}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \cdots & (a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1}, b_{m1}) & (a_{m2}, b_{m2}) & \cdots & (a_{mn}, b_{mn}) \end{bmatrix} \end{array}$$

For any nonempty subset Y_1 of $S = \{s_1, \ldots, s_m\}$ and every nonempty subset Y_2 of $T = t_1, \ldots, t_n\}$, one can check if any mixed strategy Nash equilibria $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ exist such that $Y_1 = supp(\boldsymbol{p}^*)$ and $Y_2 = supp(\boldsymbol{q}^*)$. These Nash equilibria will satisfy the 10 equations from [6] where $s^0 \in Y_1$ and $t^0 \in Y_2$ (note that s^0 and t^0 exist since Y_1 and Y_2 are nonempty) [6].

Maple Package/ Methodology

nashe composed of two procedures

nashpm- calculates all pure and mixed strategy Nash equilibria of a given bimatrix game

payoff- calculates the expected payoffs from a given pair of strategies

```
nashe := module( )
option package;
export nashpm, payoff;
```

Methods to check accuracy of solutions

- Best response analysis
- Strong domination
- Published solutions

Applications

$E := Matrix([[[e, \pi], [\sqrt{2}, \pi]], [[\sqrt{2}, e], [\pi, \sqrt{2}]]])$

$$E := \begin{bmatrix} [e, \pi] & [\sqrt{2}, \pi] \\ [\sqrt{2}, e] & [\pi, \sqrt{2}] \end{bmatrix}$$

$nashpm(E)$

$$\{p_1 = 1, p_2 = 0, q_1 = 1, q_2 = 0\}$$

$$\left\{p_1 = 1, p_2 = 0, q_1 = -q_2 + 1, q_2 \le \frac{e - \sqrt{2}}{\pi - 2\sqrt{2} + e}, 0 < q_2\right\}$$

$payoff(Vector([1, 0]), Vector([1, 0]), E),$
$payoff\left(Vector([1, 0]), Vector\left(\left[\frac{4}{5}, \frac{1}{5}\right]\right), E\right),$
$payoff\left(Vector([1, 0]), Vector\left(\left[\frac{3}{5}, \frac{2}{5}\right]\right), E\right)$

$$[e, \pi], \left[\frac{4e}{5} + \frac{\sqrt{2}}{5}, \pi\right], \left[\frac{3e}{5} + \frac{2\sqrt{2}}{5}, \pi\right]$$

$F := Matrix([[[13,-13], [29,-29], [8,-8], [12,-12], [16,-16], [23,-23]], [[18,-18], [22,-22], [21,-21], [22,-22], [29,-29], [31,-31]], [[18,-18], [22,-22], [31,-31], [31,-31], [27,-27], [37,-37]], [[11,-11], [22,-22], [12,-12], [21,-21], [21,-21], [26,-26]], [[18,-18], [16,-16], [19,-19], [14,-14], [19,-19], [28,-28]], [[23,-23], [22,-22], [19,-19], [23,-23], [30,-30], [34,-34]]])$

$$F := \begin{bmatrix} [13, -13] & [29, -29] & [8, -8] & [12, -12] & [16, -16] & [23, -23] \\ [18, -18] & [22, -22] & [21, -21] & [22, -22] & [29, -29] & [31, -31] \\ [18, -18] & [22, -22] & [31, -31] & [31, -31] & [27, -27] & [37, -37] \\ [11, -11] & [22, -22] & [12, -12] & [21, -21] & [21, -21] & [26, -26] \\ [18, -18] & [16, -16] & [19, -19] & [14, -14] & [19, -19] & [28, -28] \\ [23, -23] & [22, -22] & [19, -19] & [23, -23] & [30, -30] & [34, -34] \end{bmatrix}$$

$nashpm(F)$

$$\left\{p_1 = 0, p_2 = 0, p_3 = \frac{4}{17}, p_4 = 0, p_5 = 0, p_6 = \frac{13}{17}, q_1 = \frac{12}{17}, q_2 = 0, q_3 = \frac{5}{17}, q_4 = 0, q_5 = 0, q_6 = 0\right\}$$

$payoff\left(Vector\left(\left[0, 0, \frac{4}{17}, 0, 0, \frac{13}{17}\right]\right), Vector\left(\left[\frac{12}{17}, 0, \frac{5}{17}, 0, 0, 0\right]\right), F\right)$

$$\left[\frac{371}{17}, -\frac{371}{17}\right]$$

$assume(b < 2a, a < 2b, a > 0, b > 0);$
$G := Matrix([[[a, a], [2a, 2b]], [[2b, 2a], [b, b]]])$

$$G := \begin{bmatrix} [a\sim, a\sim] & [2a\sim, 2b\sim] \\ [2b\sim, 2a\sim] & [b\sim, b\sim] \end{bmatrix}$$

$nashpm(G)$

$$\{p_1 = 1, p_2 = 0, q_1 = 0, q_2 = 1\}$$

$$\{p_1 = 0, p_2 = 1, q_1 = 1, q_2 = 0\}$$

$$\left\{p_1 = \frac{2a\sim - b\sim}{a\sim + b\sim}, p_2 = \frac{2b\sim - a\sim}{a\sim + b\sim}, q_1 = \frac{2a\sim - b\sim}{a\sim + b\sim}, q_2 = \frac{2b\sim - a\sim}{a\sim + b\sim}\right\}$$

$payoff(Vector([1, 0]), Vector([0, 1]), G),$
$payoff(Vector([0, 1]), Vector([1, 0]), G),$
$payoff\left(Vector\left(\left[\frac{2a-b}{a+b}, \frac{2b-a}{a+b}\right]\right), Vector\left(\left[\frac{2a-b}{a+b}, \frac{2b-a}{a+b}\right]\right), G\right)$

$$[2a\sim, 2b\sim], [2b\sim, 2a\sim], \left[\frac{3a\sim b\sim}{a\sim + b\sim}, \frac{3a\sim b\sim}{a\sim + b\sim}\right]$$

Conclusion

- For $m \times n$ non-cooperative bimatrix games with real number entries, we have seen that ***nashpm*** computes all pure and mixed strategy Nash equilibria.
- For $m \times n$ non-cooperative bimatrix games with symbolic entries, ***nashpm*** will sometimes compute all pure and mixed strategy Nash equilibria.

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